

Versal deformations and superpotentials for rational curves in smooth  
threefolds

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*Dedicated to Herb Clemens on the occasion of his 60<sup>th</sup> birthday.*

## 1 Introduction.

My interests in the study of curves on Calabi-Yau threefolds have largely been shaped in two ways: by the tutelage and guidance of Herb Clemens while I was an instructor at the University of Utah in the early 80's, and by the interaction between geometry and string theory. This note is the result of both of these influences.

Herb impressed upon me the importance of the consideration of elementary examples and development of computational techniques for these examples when a general theory was not yet apparent. For example, in his own work he developed simple computational techniques for studying the moduli of immersed rational curves 10 years before Kontsevich's notion of a stable map. This led to breakthroughs in the study of the Abel-Jacobi mapping such as the infinite generation of homological equivalence modulo algebraic equivalence [4], led to his formulation of the famous Clemens conjecture on the finiteness of rational curves on the generic quintic threefold [5] and my proof in degree  $d \leq 7$  [10] which was only possible thanks to Herb's generous sharing of his ideas.

Given a curve  $C$  in any complex manifold  $M$ , there is a versal deformation space  $\mathcal{K}$  parametrizing the local moduli of deformations of  $C$  in  $M$  [12]. Letting  $N$  denote the normal bundle of  $C$  in  $M$ , the result is that there is a neighborhood  $U \subset H^0(C, N)$  containing the origin and a holomorphic obstruction map

$$F : U \rightarrow H^1(C, N)$$

with  $F(0) = 0$  and  $F'(0) = 0$  such that  $\mathcal{K}$  is defined as an analytic space by

$$\mathcal{K} = F^{-1}(0).$$

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Note that  $\mathcal{K}$  is locally completely determined by  $N$  if either  $H^0(C, N) = 0$  (in which case  $\mathcal{K}$  is a point) or  $H^1(C, N) = 0$  (in which case  $\mathcal{K}$  is smooth of dimension  $h^0(C, N)$ ).

I will focus here on the case when  $X$  is a complex threefold and  $C \simeq \mathbf{P}^1$ . By the previous paragraph, I may as well assume that  $C$  contains both deformations and obstruction spaces. In terms of the normal bundle  $N$  of  $C$  in  $X$ , this means that  $N \simeq \mathcal{O}(m) \oplus \mathcal{O}(-n)$ , with  $m \geq 0$  and  $n \geq 2$ .

In the generality considered in this note, the obstruction map can be computed explicitly.

If  $X$  is a Calabi-Yau threefold, then the versal deformation space has another description arising from string theory, since this space parametrizes D-branes wrapping holomorphic curves in  $X$ .<sup>2</sup> The result from physics is that the versal deformation space is a local description of the space of supersymmetric ground states of a 4-dimensional gauge theory with  $N = 1$  supersymmetry. Furthermore, the gauge theory contains  $h^0(C, N)$  massless chiral fields [2]. See [13] for an introduction to the physics of D-branes.

In mathematical terms, the assertion from physics implies the following description of the versal deformation space: there is a compact Lie group  $G$  acting on  $H^0(C, N)$ , together with a  $G$ -invariant holomorphic function  $W$  on a neighborhood  $U \subset H^0(C, N)$  of the origin such that

$$\mathcal{K} \simeq \text{Crit}(W)/G.$$

In physics,  $G$  arises as the gauge group, and  $W$  arises as the *superpotential*. See [8] for a description of these gauge theories written for mathematicians.

Comparing with the mathematical description of the versal deformation space, it is clear that these results can only be compatible if  $G$  acts trivially.

The main result of this note is a proof in the generality considered here that the versal deformation space is defined by the vanishing of the gradient of a single holomorphic function  $W$  on a finite dimensional space.

The superpotential  $W$  deserves to be better understood in a precise mathematical sense, as it connects several areas of geometry. In [14],  $W$  is computed as a function on the (infinite dimensional) space of  $C^\infty$  curves as an extension of the Abel-Jacobi mapping, and is shown to be holomorphic in a certain sense (see also [7]). In [9], a finite-dimensional expression for  $W$  is given in terms of holomorphic Chern-Simons theory, and its holomorphicity

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<sup>2</sup>Technically, one has to consider curves together with a flat line bundle; but if the curve is rational, then there is no distinction.

is illustrated in examples. It is an interesting problem under investigation to turn this last description into a mathematical proof that the versal deformation space is given as the critical variety of a single holomorphic function on a finite dimensional space, as is proven here in a special case.

The existence of a superpotential simplifies many calculations in deformation theory. For instance, the family of lines on quintic threefolds has received a lot of attention, and in particular, the famous 2875 lines can be accounted for by deformation away from the Fermat quintic [1]. It is shown in [3] that much of this deformation theory can be described simply by adding a perturbation term to the superpotential  $W = \rho^3 \psi^3$  for the Fermat quintic. The counting of the 2875 lines as a virtual number on the Fermat quintic has been described in [6].

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## 2 Versal deformation spaces

Let  $C \simeq \mathbf{P}^1$  be a smooth rational curve contained in a smooth complex analytic threefold  $X$ , with normal bundle  $N = \mathcal{O}(m) \oplus \mathcal{O}(-n)$ .

In this note, explicit computations will be performed for smooth rational curves  $C$  in threefolds  $X$  for which  $C$  can be covered by coordinate charts  $U_0$  and  $U_1$  of  $X$  such that:

1.  $U_0$  has coordinates  $(x, y_1, y_2)$  and  $C$  is defined in  $U_0$  by  $y_1 = y_2 = 0$ .
2.  $U_1$  has coordinates  $(w, z_1, z_2)$  and  $C$  is defined in  $U_1$  by  $z_1 = z_2 = 0$ .
3. On  $C$ , the coordinates  $w$  and  $x$  are related by  $w = x^{-1}$ .

**Remark 1** *Examples of such curves abound, for instance if  $X$  is quasi-projective and  $C$  is embedded in  $X$  as a line.*

**Remark 2** *Since considerations are purely local, it can be assumed that  $X = U_0 \cup U_1$ .*

Let  $I$  denote the ideal sheaf of  $C$  in  $X$ . Since  $I/I^2 \simeq N^* \simeq \mathcal{O}(n) \oplus \mathcal{O}(-m)$ ,  $X$  is determined locally near  $C$  modulo  $I^2$ . After a possible coordinate change, the gluing map between the patches therefore takes the form

$$\begin{aligned} z_1 &= x^n y_1 + f(x, y_1, y_2) \\ z_2 &= x^{-m} y_2 + g(x, y_1, y_2) \\ w &= x^{-1} + h(x, y_1, y_2), \end{aligned} \tag{1}$$

where  $f, g, h$  are sections of  $I^2$ , holomorphic in  $U_0 \cap U_1$  (where in (1), these sections are expressed in the coordinates of  $U_0$ ).

The goal of this section is to explicitly construct the versal deformation space of these curves, following [12]. While the main calculation is really just an exercise in adapting the more general situation in [12], a fair amount of detail will be given here, both so that this note will be more or less self-contained, and also because I think that this construction deserves to be better known.

The normal bundle  $N$  can be constructed in the usual way by gluing together trivializations over the  $V_i := U_i \cap C$ . The gluing map is given by the matrix

$$F = \begin{pmatrix} x^n & 0 \\ 0 & x^{-m} \end{pmatrix} \tag{2}$$

expressed in the coordinate  $x$  on  $V_0$ . A section of  $N$  on an analytic open set  $V \subset V_i$  will be expressed as a vector of holomorphic functions  $(\phi_1^i, \phi_2^i)$  on  $V$ . In particular, if  $\mathcal{V} = \{V_0, V_1\}$ , then the group of Čech cochains  $C^0(\mathcal{V}, N)$  can be identified with the space

$$C^0(\mathcal{V}, N) = \left\{ (\phi_1^0(x), \phi_2^0(x)), (\phi_1^1(w), \phi_2^1(w)) \right\},$$

of pairs of vector valued-holomorphic functions in  $V_0$  (resp.  $V_1$ ). The convention used here is that the two sections of  $N$  defined above agree on  $V_0 \cap V_1$  if

$$(\phi_1^1(w), \phi_2^1(w)) = (\phi_1^0(x), \phi_2^0(x)) F, \tag{3}$$

where  $F$  is the transition matrix (2). It follows that the vector space  $H^0(N) \subset C^0(\mathcal{V}, N)$  is given as the space of cocycles

$$\left\{ \left( \left( 0, \sum_{i=0}^m a_i x^i \right), \left( 0, \sum_{i=0}^m a_i w^{m-i} \right) \right) \right\}. \tag{4}$$

The  $a_i$  serve as coordinates on  $H^0(N)$ , and the versal deformation space will be given explicitly by equations in the  $a_i$ .

In addition,  $H^1(N)$  can be given by Čech cohomology with respect to the cover  $\mathcal{V}$ . There is the usual coboundary map  $\delta : C^0(\mathcal{V}, N) \rightarrow C^1(\mathcal{V}, N)$ . I fix the convention throughout that in writing elements of  $C^1(\mathcal{V}, N)$ , i.e. sections of  $N$  over  $V_0 \cap V_1$ , the trivialization of  $N$  over  $V_1$  will be used. With this convention, the coboundaries are just the sections whose series expansions have the form

$$\left( \sum_{i \leq -n, i \geq 0} b_i w^i, \sum_i c_i w^i \right).$$

The result is that

$$H^1(N) \simeq \left\{ \left( \sum_{i=1}^{n-1} b_{-i} w^{-i}, 0 \right) \right\}. \quad (5)$$

There is also an obvious projection  $H : C^1(\mathcal{V}, N) \rightarrow H^1(N)$  defined by the natural truncation of  $\sum_i b_i w^i$ :

$$H \left( \sum_i b_i w^i, \sum_i c_i w^i \right) = \left( \sum_{i=1}^{n-1} b_{-i} w^{-i}, 0 \right). \quad (6)$$

Following [12], define the map

$$K : C^0(\mathcal{V}, N) \rightarrow C^1(\mathcal{V}, N)$$

by putting

$$\begin{aligned} K((\phi_1^0(x), \phi_2^0(x)), (\phi_1^1(w), \phi_2^1(w))) = \\ (\phi_1^1(x^{-1} + h(x, \phi_1^0(x), \phi_2^0(x)) - (x^n \phi_1^0(x) + f(x, \phi_1^0(x), \phi_2^0(x))), \\ \phi_2^1(x^{-1} + h(x, \phi_1^0(x), \phi_2^0(x)) - (x^{-m} \phi_2^0(x) + g(x, \phi_1^0(x), \phi_2^0(x)))). \end{aligned} \quad (7)$$

Note that  $K$  has been chosen to have the following property: the equations

$$y_j = \phi_j^0(x), \quad z_k = \phi_k^1(w), \quad w = x^{-1} + h(x, \phi_1^0(x), \phi_2^0(x))$$

patch to define a compact complex curve if and only if

$$K((\phi_1^0(x), \phi_2^0(x)), (\phi_1^1(w), \phi_2^1(w))) = 0.^3$$

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<sup>3</sup>For this to make sense, a norm is needed on  $C^0(\mathcal{V}, N)$  and the cochain  $(\phi_1^0(x), \phi_2^0(x)), (\phi_1^1(w), \phi_2^1(w))$  needs to be kept sufficiently small to ensure that this curve actually makes sense in  $X$ . As there is little risk of confusion, I will continue this imprecise practice throughout and defer to [12] for the necessary estimates. In this way, calculations can be performed using series expansions without having to worry about convergence issues.

Continuing to adapt to the notation of [12], choose the projection

$$B : C^1(\mathcal{V}, N) \rightarrow \delta C^0(\mathcal{V}, N)$$

defined by

$$B \left( \sum_i b_i w^i, \sum_i c_i w^i \right) = \left( \sum_{i \leq -n, i \geq 0} b_i w^i, \sum_i c_i w^i \right) \quad (8)$$

and the mapping

$$E_0 : \delta C^0(\mathcal{V}, N) \rightarrow C^0(\mathcal{V}, N)$$

given by

$$E_0 \left( \sum_{i \leq -n, i \geq 0} b_i w^i, \sum_i c_i w^i \right) = \left( (-\sum_{i \leq -n} b_i x^{-n-i}, -\sum_{i < 0} c_i x^{m-i}), (\sum_{i \geq 0} b_i w^i, \sum_{i \geq 0} c_i w^i) \right), \quad (9)$$

which satisfies  $\delta E_0 = 1$ .

Finally define  $L : C^0(\mathcal{V}, N) \rightarrow C^0(\mathcal{V}, N)$  by

$$L(\phi) = \phi + E_0 B K \phi - E_0 \delta \phi. \quad (10)$$

Put

$$M = \{\phi \mid K(\phi) = 0\}. \quad (11)$$

As described above,  $M$  parametrizes deformations of  $C$ , but is way too big to be a versal deformation space. In particular, it needs to be cut down so that its tangent space is  $H^0(N)$ .

**Lemma 1** (Namba)  *$L$  is invertible in a neighborhood of 0, and  $L(M) \subset H^0(N)$ .*

*Proof:* For the first statement, compute  $L'(0) = 1 + E_0 B \delta - E_0 \delta = 1$ . For the second statement, compute that if  $\phi \in M$ , then  $\delta L(\phi) = \delta \phi - \delta E_0 \delta \phi = \delta \phi - \delta \phi = 0$ .  $\square$

Now choose a sufficient small neighborhood  $U$  of  $0 \in H^0(N)$ , and write elements of  $U$  as in (4)

$$s(a) = \left( \left( 0, \sum_{i=0}^m a_i x^i \right), \left( 0, \sum_{i=0}^m a_i w^{m-i} \right) \right) \quad (12)$$

with  $a = (a_0, \dots, a_m)$  constrained to a suitable neighborhood of 0. Then write

$$HKL^{-1} \left( (0, \sum_{i=0}^m a_i x^i), (0, \sum_{i=0}^m a_i w^{m-i}) \right) = \left( \sum_{i=1}^{n-1} k_i(a_0, \dots, a_m) w^{-i}, 0 \right),$$

where  $H, K$  and  $L$  are respectively given by (6), (7), and (10). The main result is

**Proposition 1** *The versal deformation space  $\mathcal{K}$  is the analytic space defined in  $U$  by the vanishing of the  $k_i(a_0, \dots, a_m)$ ,  $1 \leq i \leq n-1$ .*

*Proof:* [12]. □

The construction of  $\mathcal{K}$  simplifies somewhat and can be made even more explicit if  $f = f(x, y_2)$  is independent of  $y_1$  and is holomorphic in all of  $U_0$ , while  $g = h = 0$ . This is in fact the situation occurring in the examples in Section 2 of [11]. Such curves will be called *Laufer curves*.

In this situation,  $L$  is more explicitly given by

$$L(\phi) = \phi + E_0 B \left( (-f(x, \phi_2^0), 0) \right)$$

which is easy to invert as follows.

Consider the function

$$g(x, a_0, \dots, a_m) = f(x, \sum_{i=0}^m a_i x^i). \tag{13}$$

It has a Taylor expansion

$$g = \sum_{i=0}^{\infty} f_i(a_0, \dots, a_m) x^i,$$

where the  $f_i$  are holomorphic in the  $a_i$ . Put

$$h(x, a_0, \dots, a_m) = \sum_{i=n}^{\infty} f_i(a_0, \dots, a_m) x^{i-n}.$$

Then compute

$$L \left( (-h, \sum_i a_i x^i), (k_0(a_0, \dots, a_m), \sum a_i w^{m-i}) \right) = s(a),$$

where  $s(a)$  is the section (12) of  $N$ . This means that

$$L^{-1}(s(a)) = \left( (-h, \sum_i a_i x^i), (f_0(a_0, \dots, a_m), \sum a_i w^{m-i}) \right).$$

Direct computation yields

$$HKL^{-1} \left( \left( 0, \sum_{i=0}^m a_i x^i \right), \left( 0, \sum_{i=0}^m a_i w^{m-i} \right) \right) = \left( -\sum_{i=1}^{n-1} f_i(a_0, \dots, a_m) x^i, 0 \right).$$

As in the general case, the versal deformation space  $\mathcal{K}$  of  $C$  is defined by the equations

$$k_1(a_0, \dots, a_m) = \dots = k_{n-1}(a_0, \dots, a_m) = 0, \quad k_i = -f_i. \quad (14)$$

In this situation, it is easy to construct a universal family of curves  $\mathcal{C}$  over  $\mathcal{K}$  explicitly. The family  $\mathcal{C} \subset \mathcal{K} \times X$  is given locally by two equations. In the patch,  $\mathcal{K} \times U_0$ ,  $\mathcal{C}$  is defined by

$$y_1 = -\sum_{j=n}^{\infty} k_j(a_0, \dots, a_m) x^{j-n}, \quad y_2 = \sum_{i=0}^m a_i x^i,$$

and in the patch  $\mathcal{K} \times U_1$ ,  $\mathcal{C}$  is defined by

$$z_1 = k_0(a_0, \dots, a_m), \quad z_2 = \sum_{i=0}^m a_i w^{m-i}.$$

To see that we obtain a family  $\mathcal{C}$ , we only have to check that these glue via (1). The equation for  $z_2$  is obvious. In the right hand side of the equation for  $z$ , we get  $\sum_{i=0}^{n-1} k_i(a_0, \dots, a_m) x^i$ , which simplifies to  $k_0(a_0, \dots, a_m)$  when the equations defining  $\mathcal{K}$  are used.

**Example 1.** This example is from [11].

$$\begin{aligned} z_1 &= x^3 y_1 + y_2^2 + x^2 y_2^{2n+1} \\ z_2 &= x^{-1} y_2 \end{aligned}$$

This is the famous example of a contractible curve with normal bundle  $\mathcal{O}(1) \oplus \mathcal{O}(-3)$ . Compute from (13)

$$g(x, a_0, a_1) = (a_0 + a_1 x)^2 + x^2 (a_0 + a_1 x)^{2n+1}.$$

Equation (14) says to extract the coefficients of  $x$  and  $x^2$  and include a minus sign, giving

$$k_1(a_0, a_1) = -2a_0a_1, \quad k_2(a_0, a_1) = -(a_1^2 + a_0^{2n+1}),$$

so the versal deformation space is defined by  $k_1 = k_2 = 0$ . Note that the versal deformation space is therefore concentrated at the origin. In fact,  $C$  is contractible [11] and therefore does not deform.

Before stating and proving the main result, a trivial lemma is useful.

**Lemma 2** *Consider any analytic function of  $x$  and the  $a_i$  of the form*

$$h(x, a_0, \dots, a_m) = r\left(x, \sum_{i=0}^m a_i x^i\right),$$

where  $r = r(x, y)$  is analytic in  $(x, y)$ . Write  $h = \sum_{i=0}^{\infty} h_i(a_0, \dots, a_m) x^i$ . Then for all  $i, j, k$  such that  $0 \leq j \leq m$  and  $0 \leq k + j - i \leq m$ ,

$$\frac{\partial h_i}{\partial a_j} = \frac{\partial h_k}{\partial a_{k+j-i}}$$

*Proof:* Compute

$$\frac{\partial h}{\partial a_j} = x^j \frac{\partial g}{\partial y} \left( x, \sum_{i=0}^m a_i x^i \right). \quad (15)$$

The desired result follows immediately by comparing terms in the series expansions of  $\partial h / \partial a_j$  and  $\partial h / \partial a_{k+j-i}$ .  $\square$

Now suppose that  $X$  has trivial canonical bundle, or equivalently, that  $m - n = -2$ . The versal deformation space is described by  $m + 1$  variables  $a_0, \dots, a_m$ , and is defined by  $n - 1$  equations. This restriction on the canonical bundle implies that there are as many equations as unknowns in (14).

**Proposition 2** *Suppose that  $C \subset X$  is a Laufer curve in a threefold  $X$  with trivial canonical bundle. Describe the versal deformation space  $\mathcal{K}$  as an analytic space inside a neighborhood  $U$  of  $0$  in  $H^0(N)$  as in (14). Then there exists a single holomorphic function  $W$  on  $U$  such that  $\mathcal{K}$  is defined as an analytic space as the subvariety of critical points of  $W$ .*

*Proof:* More precisely,  $W$  will be chosen to satisfy

$$\frac{\partial W}{\partial a_i} = k_{n-1-i}.$$

Since the claim is local, it suffices to check the integrability condition

$$\frac{\partial k_{n-1-i}}{\partial a_j} = \frac{\partial k_{n-1-j}}{\partial a_i}.$$

But this follows immediately from the form of  $k_i$  and the Lemma.  $\square$

In the Example 1, note that

$$(k_2, k_1) = -\text{grad} \left( a_0 a_1^2 + \frac{a_0^{2n+2}}{2n+2} \right),$$

so we can take as the superpotential

$$W = - \left( a_0 a_1^2 + \frac{a_0^{2n+2}}{2n+2} \right).$$

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